1. Historical Control Systems

Feedback by System Design

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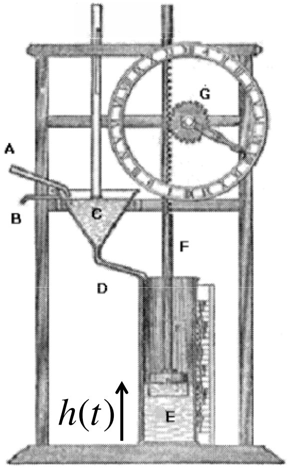
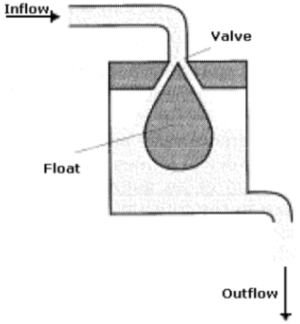
Flow Regulated Water Clock (250BC)

- 250 BC flow regulated water clock
  - Ctesibius, a Greek Inventor

Constant flow rate

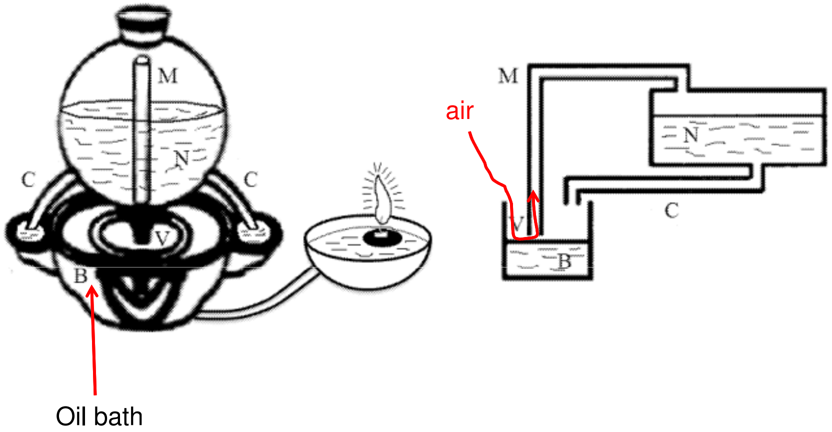
$$h(t)=\frac{V(t)}{A}=\frac{Ft}{A}=Kt$$





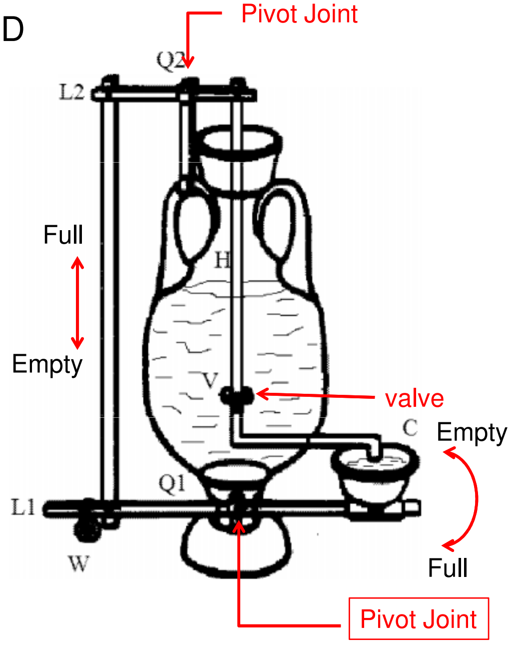
Self Re-filling Oil Lamp (200 BC)

- Self Re-filling mechanism
  - Philon, a Greek inventor



Weight Regulated Liquid Filling Device

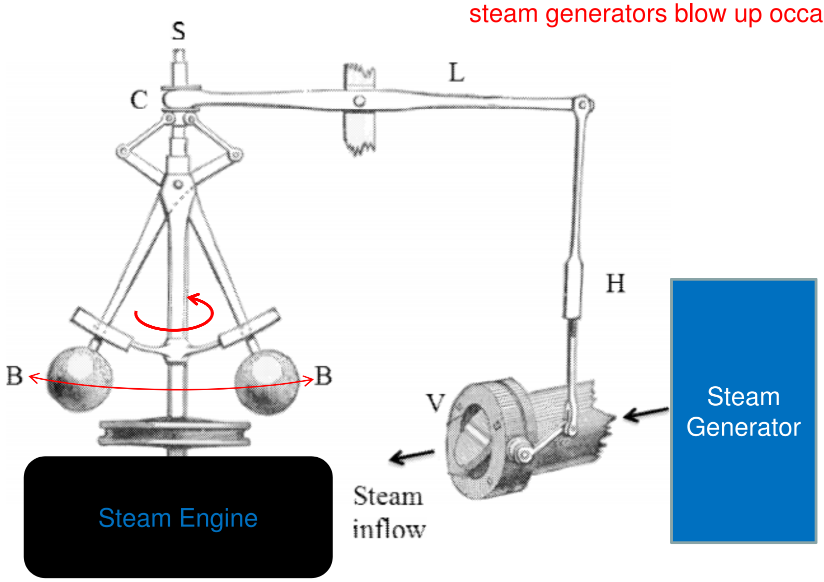
- 1st Century AD



Flyball Governor (1788)

- James Watt (1788)

First crude Governors were working well. Precisely machined governors caused steam generators blow up occasionally.



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2. Plant Model and Response

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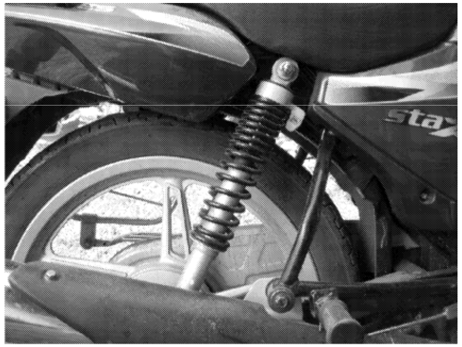
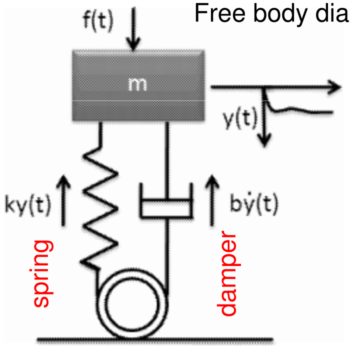
Plant Model: Mechanical System

- Shock absorber



Free body diagram



- Spring resists displacement
- Damper resists speed

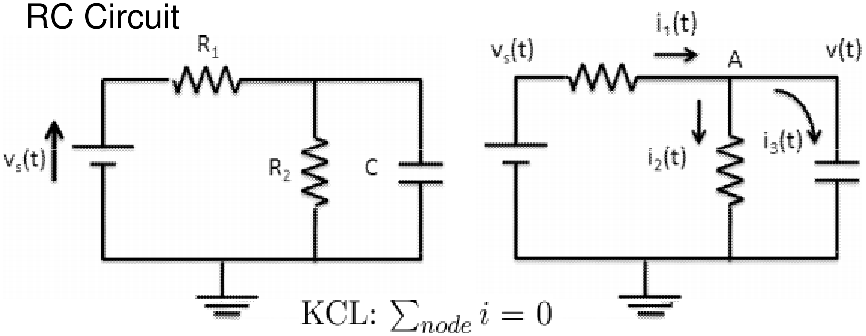
$$m\ddot{y}(t) = f - ky(t) - b\dot{y}(t) \quad (2.1)$$

$$\ddot{y}(t) + \frac{b}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{1}{m}f(t) \quad (2.2)$$

Second order model

Plant Model: Electrical System

- RC Circuit



$$\text{KCL: } \sum_{node} i = 0$$

$$\begin{aligned} i_2(t) + i_3(t) &= i_1(t) \\ C\dot{v}(t) + \frac{1}{R_2}v(t) &= \frac{v_s(t) - v(t)}{R_1} \\ R_1R_2C\dot{v}(t) + R_1v(t) &= R_2(v_s(t) - v(t)) \\ R_1R_2C\dot{v}(t) + (R_1 + R_2)v(t) &= R_2v_s(t) \end{aligned}$$

Plant Model: Electrical System

$$\begin{aligned}\dot{v}(t) + \frac{R_1+R_2}{R_1R_2C}v(t) &= \frac{1}{R_1C}v_s(t)\\ \dot{v}(t) + av(t) &= bv_s(t)\end{aligned}$$

(2.3)

First Order System

$$a = \frac{R_1+R_2}{R_1R_2C} \qquad b = \frac{1}{R_1C}$$

- System models are ordinary differential equations (ODEs).
- The order of model ODE depends on the system complexity.
- The response (output) of the plant can be obtained by solving model ODE for a given forcing function (input)
- Laplace transforms can be used to solve ODEs efficiently

Laplace Transforms

function	$f(t)$	$F(s)$
unit impulse	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
time exponent	e^{at}	$\frac{1}{s-a}$
cosin	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
power of time	t^n	$\frac{n!}{s^{n+1}}$
linearity	$\alpha_1f_1(t) \pm \alpha_2f_2(t)$	$\alpha_1F_1(s) \pm \alpha_2f_2(s)$
exponential scaling	$e^{at}f(t)$	$F(s-a)$
time shift	$f(t \pm T)$	$e^{\pm sT}F(s)$
time multiplication	$tf(t)$	$-\frac{d}{ds}F(s)$
differential	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)$ $\cdots - f^{n-1}(0)$
integral	$\int f(t)dt$	$\frac{1}{s}F(s)$
time scaling	$f(at)$	$\frac{1}{a}e^{\frac{1}{a}}F(s)$
convolution integral	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$

System Response with Laplace

- RC Circuit Model $\dot{v}(t) + av(t) = bv_s(t)$ (2.3)
  - When transformed into Laplace domain L{ } (forward)

$$\begin{aligned}sV(s) - v(0) + aV(s) &= bV\frac{1}{s}\\ (s+a)V(s) &= v(0) + bV\frac{1}{s}\\ V(s) &= \frac{1}{s+a}v(0) + \frac{b}{s(s+a)}V\end{aligned}$$

(3.47)

For DC Voltage

$v_s(t) = Vu_s(t)$

$$V(s) = \frac{1}{s+a}v(0) + \frac{b}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)V$$

Method 1: Partial fraction

- Response
  - When transformed back to time domain L⁻¹{ } (inverse)

$$v(t) = v(0)e^{-at} + \frac{b}{a}V\left(1 - e^{-at}\right)$$

(3.49)

System Response with Laplace

(3.47), $$V(s) = \frac{1}{s+a}v(0) + \frac{b}{s(s+a)}V$$

$u(t)$ e^{-at}

Method 2:
Convolution Integral

- Transforming back to time domain

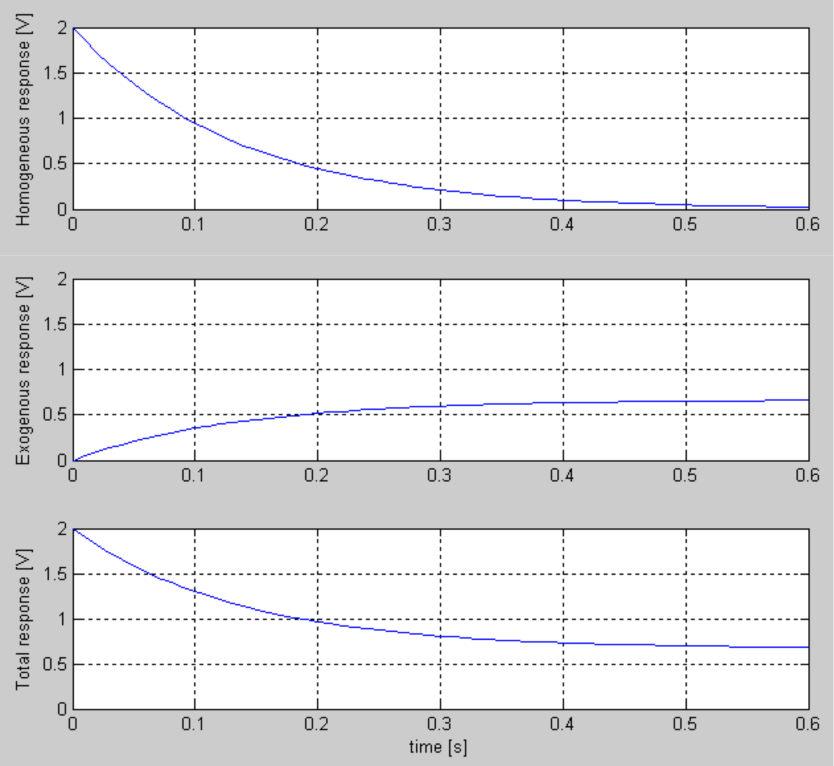
$$\begin{aligned}v(t) &= v(0)e^{-at} + bV_s\int_0^t e^{-a\tau}u_s(t-\tau)d\tau\\ &= v(0)e^{-at} + bV_s\int_0^t e^{-a\tau}.1d\tau\\ &= v(0)e^{-at} + \frac{-b}{a}V_se^{-a\tau}|_0^t\\ &= \underbrace{v(0)e^{-at}}_{\text{Homogeneous Response}} + \underbrace{\frac{b}{a}V_s\left(1 - e^{-at}\right)}_{\text{Exogenous Response}}\end{aligned}$$

(3.50)

Matlab Simulation: RC Circuit

```
%% First Order Response: RC Circuit
R1=1000; R2=2000; C=200*10^-6;        % circuit components
a=(R1+R2)/(R1*R2*C); b=1/(R1*C);      % model coefficients
v0=2;                                 % initial condition
dur=0.6                               % simulation duration
t=[0:0.01:dur];

%% Homogeneous Response
yH=v0*exp(-a*t);

%% Exogenous Response
yE=(b/a)*(1-exp(-a*t));

%% Total Response
yT=yH+yE;

%% Plot graphs
subplot(311); plot(t,yH); axis([0 dur 0 2]);
ylabel('Homogeneous response [V]'); grid on;

subplot(312); plot(t,yE); axis([0 dur 0 2]);
ylabel('Exogenous response [V]'); grid on;

subplot(313); plot(t,yT); axis([0 dur 0 2]);
ylabel('Total response [V]'); xlabel('time [s]'); grid on;
```

Matlab Simulation: Step Response



System Response with Laplace

- Mechanical System Response

(2.2) $$\ddot{y}(t) + \frac{b}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{1}{m}f(t)$$

$$\ddot{y}(t) + 2\sigma\dot{y}(t) + \rho y(t) = \eta f(t) \quad (3.51)$$

Where

$$2\sigma = \frac{b}{m},\ \rho = \frac{k}{m},\ \text{and}\ \eta = \frac{1}{m}$$

- Transforming into Laplace domain

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 2\sigma[sY(s) - y(0)] + \rho Y(s) &= \eta F(s) \\ (s^2 + 2\sigma s + \rho)Y(s) - y(0)s - [2\sigma y(0) + y'(0)] &= \eta F(s) \end{aligned}$$

$$Y(s) = \frac{y(0)s + [2\sigma y(0) + y'(0)]}{(s^2 + 2\sigma s + \rho)} + \frac{\eta}{(s^2 + 2\sigma s + \rho)}F(s) \qquad (3.52)$$

- Three possible scenarios based on the solutions of the denominator polynomial (characteristic equation)

$$(2\sigma)^2 - 4.1.\rho = \sigma^2 - \rho \longleftarrow \quad \Delta(s) = s^2 + 2\sigma s + \rho = 0$$

Determinant

System Response : Partial Fractions

- *Case 1:* $\sigma^2 - \rho > 0\ (b > 2\sqrt{mk})$

$K_1 = y(0)$, and $K_2 = 2\sigma y(0) + y'(0)$

(3.52) $$Y(s) = \frac{K_1 s + K_2}{(s-\alpha_1)(s-\alpha_2)} + \frac{\eta}{(s-\alpha_1)(s-\alpha_2)}F(s) \longleftarrow F(s) = \frac{A}{s}$$

where $\alpha_1, \alpha_2 = -\sigma \pm \sqrt{\sigma^2 - \rho}$ **negative, real, distinct poles**

Poles are determined by the system parameters

- *Case 1: Partial Fractioning*

$$\begin{aligned} Y(s) &= \frac{K_1 s + K_2}{(s-\alpha_1)(s-\alpha_2)} + \frac{\eta}{(s-\alpha_1)(s-\alpha_2)}\frac{A}{s} \qquad (3.55) \\ &= \frac{P_1}{(s-\alpha_1)} + \frac{P_2}{(s-\alpha_2)} + \eta A\left(\frac{Q_1}{(s-\alpha_1)} + \frac{Q_2}{(s-\alpha_2)} + \frac{Q_3}{s}\right) \end{aligned}$$

where $$\underbrace{P_1 = \frac{K_1\alpha_1 + K_2}{\alpha_1 - \alpha_2}, P_2 = \frac{K_1\alpha_2 + K_2}{\alpha_2 - \alpha_1},}_{\text{From initial conditions and system parameters}} \underbrace{Q_1 = \frac{1}{\alpha_1(\alpha_1 - \alpha_2)}, Q_2 = \frac{1}{\alpha_2(\alpha_2 - \alpha_1)}, Q_3 = \frac{1}{\alpha_1\alpha_2}}_{\text{From system parameters}}$$

System Response : Convolution Integral

- *Transforming back to time domain*

$$\begin{aligned} y(t) &= P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} + \eta A\left(Q_1 e^{\alpha_1 t} + Q_2 e^{\alpha_2 t}\right) + \eta A Q_3 \\ &= \eta A Q_3 + \underbrace{(P_1 + \eta A Q_1)e^{\alpha_1 t} + (P_2 + \eta A Q_2)e^{\alpha_2 t}}_{\text{Transient response}} \qquad (3.56) \end{aligned}$$

Steady-state response

Decay with time

- *Case 1: Convolution Integral Method*

  Using coverup method (see Appendix)

(3.55),
$$\begin{aligned} Y(s) &= \left\{\frac{P_1}{(s-\alpha_1)} + \frac{P_2}{(s-\alpha_2)}\right\} + \eta\left\{\frac{P_3}{(s-\alpha_1)} + \frac{P_4}{(s-\alpha_2)}\right\}F(s) \\ &= \frac{P_1}{(s-\alpha_1)} + \frac{P_2}{(s-\alpha_2)} + \eta\frac{P_3}{(s-\alpha_1)}F(s) + \eta\frac{P_4}{(s-\alpha_2)}F(s) (3.57) \end{aligned}$$

where $P_3 = \frac{1}{\alpha_1 - \alpha_2}$ $P_4 = \frac{1}{\alpha_2 - \alpha_1}$

System Response with Laplace

- *Transforming back to time domain*

$$y(t) = P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} + \eta P_3 \int_0^t e^{\alpha_1 (t-\tau)} f(\tau) d\tau + \eta P_4 \int_0^t e^{\alpha_2 (t-\tau)} f(\tau) d\tau$$

- *For* $f(t) = Au_s(t)$

$$\begin{aligned} y(t) &= P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} + \eta P_3 \int_0^t e^{\alpha_1 (t-\tau)} A d\tau + \eta P_4 \int_0^t e^{\alpha_2 (t-\tau)} A d\tau \\ &= P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} + \eta P_3 e^{\alpha_1 t} \int_0^t e^{-\alpha_1 \tau} A d\tau + \eta P_4 e^{\alpha_2 t} \int_0^t e^{-\alpha_2 \tau} A d\tau \\ &= P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} - \frac{\eta P_3 A e^{\alpha_1 t}}{\alpha_1} e^{-\alpha_1 \tau}\Big|_0^t - \frac{\eta P_4 A e^{\alpha_2 t}}{\alpha_2} e^{-\alpha_2 \tau}\Big|_0^t \\ &= P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} - \frac{\eta P_3 A e^{\alpha_1 t}}{\alpha_1}(e^{-\alpha_1 t} - 1) - \frac{\eta P_4 e^{\alpha_2 t} A}{\alpha_2}(e^{-\alpha_2 t} - 1) \\ &= P_1 e^{\alpha_1 t} + P_2 e^{\alpha_2 t} + \frac{\eta P_3 A}{\alpha_1}(1 - e^{\alpha_1 t}) + \frac{\eta P_4 A}{\alpha_2}(1 - e^{\alpha_2 t}) \\ &= \underbrace{-\eta A\left(\frac{P_3}{\alpha_1} + \frac{P_4}{\alpha_2}\right)}_{\text{Steady-state response}} + \underbrace{\left(P_1 + \frac{\eta A P_3}{\alpha_1}\right)e^{\alpha_1 t} + \left(P_2 + \frac{\eta A P_4}{\alpha_2}\right)e^{\alpha_2 t}}_{\text{Transient response}} \end{aligned}$$

(3.59)

Steady State and Transient Responses

- Steady State Response
  - The sustainable response over time
  - Depends only on external forcing function

$$-\eta A\left(\frac{P_3}{\alpha_1} + \frac{P_4}{\alpha_2}\right)$$

- Transient response
  - Decaying response
  - Depends both on Initial conditions and external forcing function

$$\left(P_1 + \frac{\eta A P_3}{\alpha_1}\right)e^{\alpha_1 t} + \left(P_2 + \frac{\eta A P_4}{\alpha_2}\right)e^{\alpha_2 t}$$

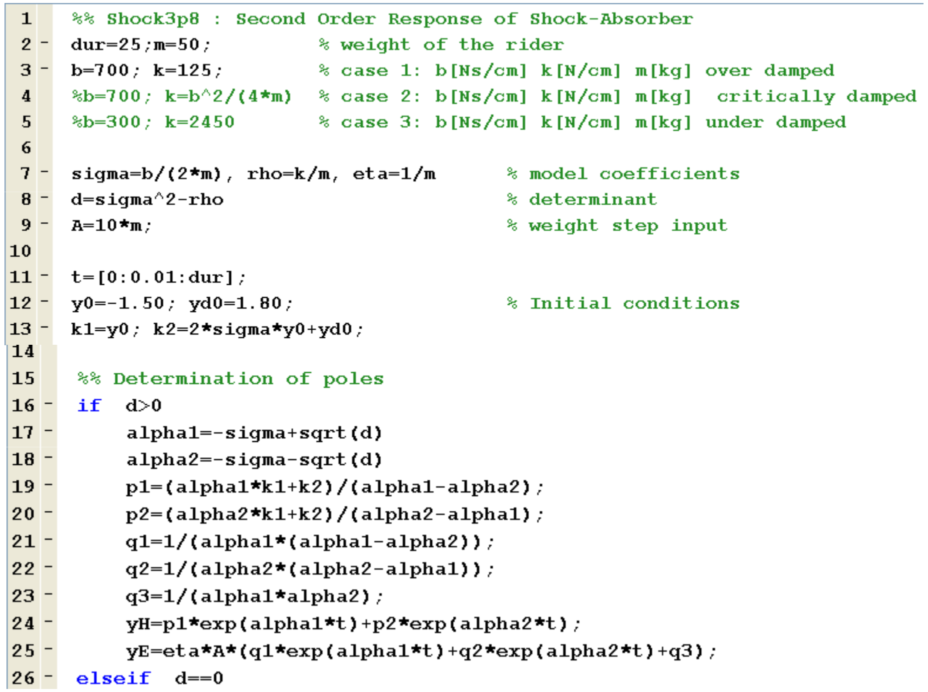
Simulation:
Case 1 Over Damped Shock-Absorber

- Set parameters as follows $k = 125\text{N/cm}$ $b = 700\text{Ns/cm}$
  - Damper is stronger than spring action
  - Speed is strongly opposed

- Then, from (3.51) $\sigma = 7,\ \rho = 2.5,\ \text{and}\ \eta = 0.2$
- Consequently, the two system poles are $\alpha_1 = \text{-}0.181\ \text{and}\ \alpha_2 = \text{-}13.819$

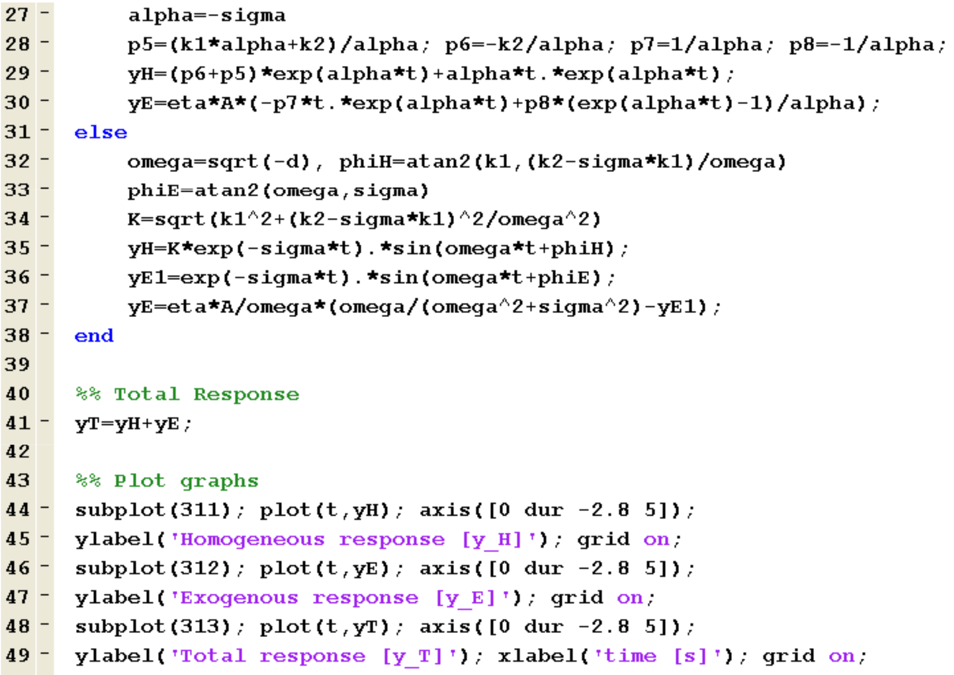
-ve real distinct poles

Matlab Code



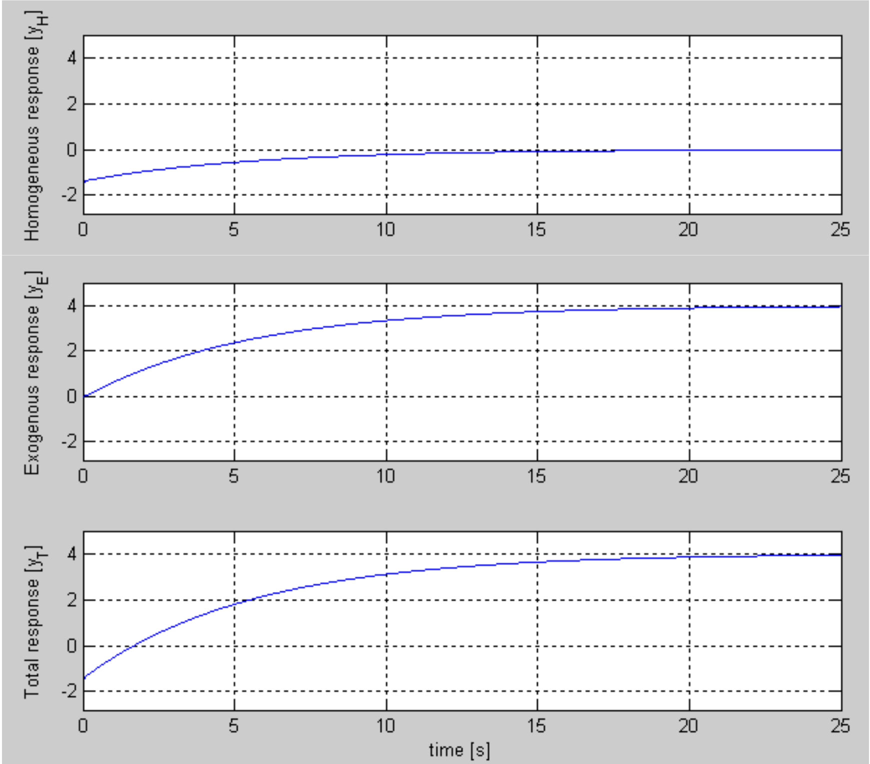
```
%% Shock3p8 : Second Order Response of Shock-Absorber
dur=25;m=50;            % weight of the rider
b=700; k=125;           % case 1: b[Ns/cm] k[N/cm] m[kg] over damped
%b=700; k=b^2/(4*m)     % case 2: b[Ns/cm] k[N/cm] m[kg]  critically damped
%b=300; k=2450          % case 3: b[Ns/cm] k[N/cm] m[kg] under damped

sigma=b/(2*m), rho=k/m, eta=1/m       % model coefficients
d=sigma^2-rho                         % determinant
A=10*m;                               % weight step input

t=[0:0.01:dur];
y0=-1.50; yd0=1.80;                   % Initial conditions
k1=y0; k2=2*sigma*y0+yd0;

%% Determination of poles
if  d>0
    alpha1=-sigma+sqrt(d)
    alpha2=-sigma-sqrt(d)
    p1=(alpha1*k1+k2)/(alpha1-alpha2);
    p2=(alpha2*k1+k2)/(alpha2-alpha1);
    q1=1/(alpha1*(alpha1-alpha2));
    q2=1/(alpha2*(alpha2-alpha1));
    q3=1/(alpha1*alpha2);
    yH=p1*exp(alpha1*t)+p2*exp(alpha2*t);
    yE=eta*A*(q1*exp(alpha1*t)+q2*exp(alpha2*t)+q3);
elseif  d==0
```

Matlab Code



```
    alpha=-sigma
    p5=(k1*alpha+k2)/alpha; p6=-k2/alpha; p7=1/alpha; p8=-1/alpha;
    yH=(p6+p5)*exp(alpha*t)+alpha*t.*exp(alpha*t);
    yE=eta*A*(-p7*t.*exp(alpha*t)+p8*(exp(alpha*t)-1)/alpha);
else
    omega=sqrt(-d), phiH=atan2(k1,(k2-sigma*k1)/omega)
    phiE=atan2(omega,sigma)
    K=sqrt(k1^2+(k2-sigma*k1)^2/omega^2)
    yH=K*exp(-sigma*t).*sin(omega*t+phiH);
    yE1=exp(-sigma*t).*sin(omega*t+phiE);
    yE=eta*A/omega*(omega/(omega^2+sigma^2)-yE1);
end

%% Total Response
yT=yH+yE;

%% Plot graphs
subplot(311); plot(t,yH); axis([0 dur -2.8 5]);
ylabel('Homogeneous response [y_H]'); grid on;
subplot(312); plot(t,yE); axis([0 dur -2.8 5]);
ylabel('Exogenous response [y_E]'); grid on;
subplot(313); plot(t,yT); axis([0 dur -2.8 5]);
ylabel('Total response [y_T]'); xlabel('time [s]'); grid on;
```

Simulation: Over Damped



System Response

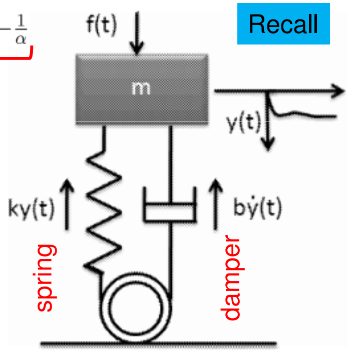
- *Case 2:* $\sigma^2 - \rho = 0,\ (b = 2\sqrt{mk})$ → Real, -ve coincident poles

(3.52) →
$$Y(s) = \frac{K_1 s + K_2}{(s-\alpha)^2} + \frac{\eta}{(s-\alpha)^2} F(s)$$

$$Y(s) = \left\{ \frac{P_5 s}{(s-\alpha)^2} + \frac{P_6}{(s-\alpha)} \right\} + \eta \left\{ \frac{P_7 s}{(s-\alpha)^2} + \frac{P_8}{(s-\alpha)} \right\} F(s) \qquad (3.61)$$

where $\underbrace{P_5 = \frac{K_1\alpha + K_2}{\alpha},\ P_6 = -\frac{K_2}{\alpha},}_{\text{IC and System}}\ \underbrace{P_7 = \frac{1}{\alpha},\ \text{and}\ P_8 = -\frac{1}{\alpha}}_{\text{System}}$

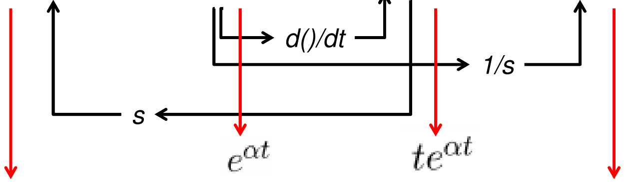
Recall



System Response: *Case 2 Critical Damping*

For $f(t) = Au_s(t) \longrightarrow F(s) = A\frac{1}{s}$

$$\begin{aligned} Y(s) &= \left\{ \frac{P_5 s}{(s-\alpha)^2} + \frac{P_6}{(s-\alpha)} \right\} + \eta \left\{ \frac{P_7 s}{(s-\alpha)^2} + \frac{P_8}{(s-\alpha)} \right\} A\frac{1}{s} \quad (3.62) \\ &= P_5 \frac{s}{(s-\alpha)^2} + P_6 \frac{1}{(s-\alpha)} \eta A P_7 \frac{1}{(s-\alpha)^2} + \eta A P_8 \frac{1}{s(s-\alpha)} \end{aligned}$$



$\frac{d(te^{\alpha t})}{dt} + te^{\alpha t}|_{t=0} = (1+\alpha t)e^{\alpha t}$ $\int_0^t e^{\alpha t} dt = \frac{1}{\alpha} e^{\alpha t}|_0^t = \frac{1}{\alpha}(e^{\alpha t} - 1)$

Responce

$$\begin{aligned} y(t) &= P_5(1+\alpha t)e^{\alpha t} + P_6 e^{\alpha t} + \eta A P_7 t e^{\alpha t} + \eta A P_8 \frac{1}{\alpha}(e^{\alpha t} - 1) \\ &= -\frac{\eta A P_8}{\alpha} + \left(\frac{\eta A P_8}{\alpha} + P_6 + P_5 \right) e^{\alpha t} + (\eta A P_7 + P_5 \alpha) t e^{\alpha t} (3.63) \end{aligned}$$

Simulation: *Case 2 Critical Damping*

Increase Spring Constant

$$\begin{tikzcd} k=125\text{N/cm} \arrow[d] & b=700\text{Ns/cm} \arrow[d] \\ k\ =2450[\text{N/cm}] \arrow[d] & k = \frac{b^2}{2m} \arrow[l, red] \\ f/k\ =\ 500/2450 = 0.2cm & \end{tikzcd}$$

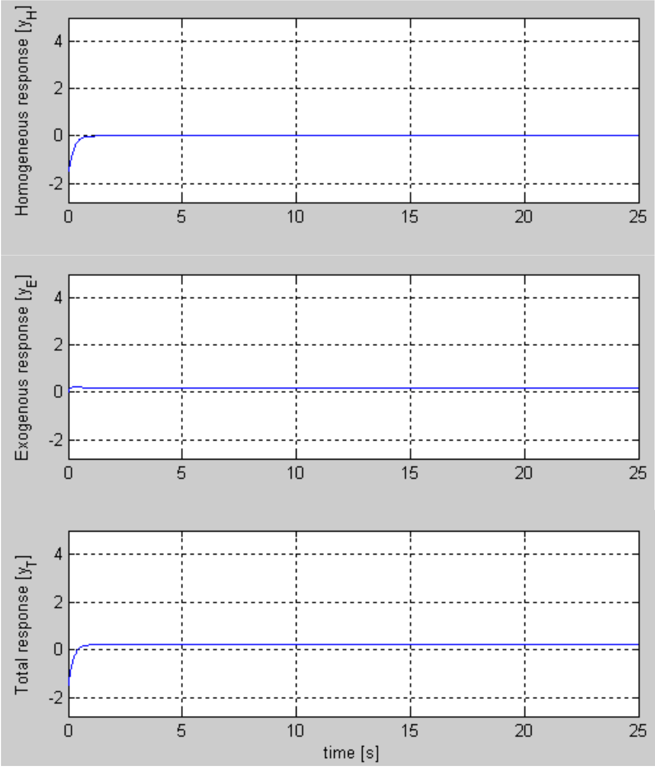
Deflection at steady state

Poles $\alpha_1 = \alpha_2 = \alpha = -7$

MatLab Simulation

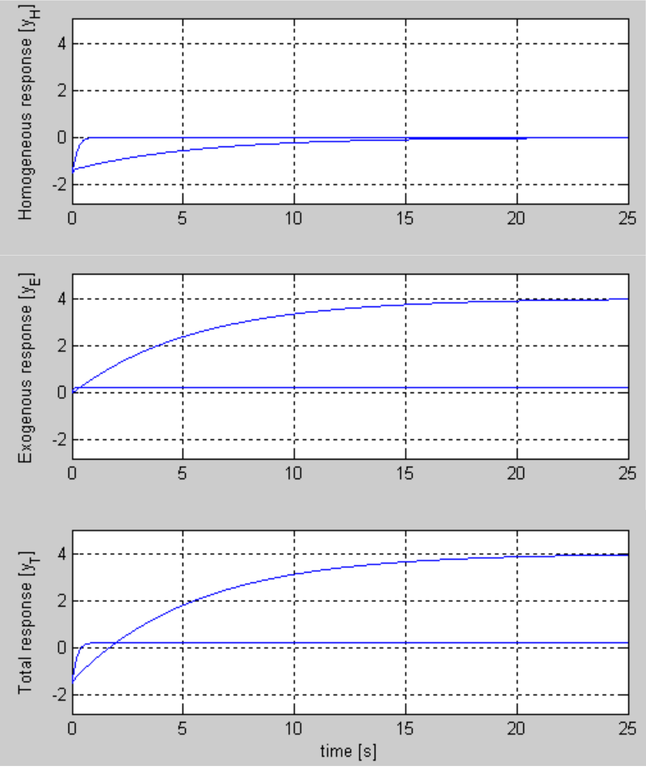
Critical Damping

- Fast response
- No overshoots
- Most energy efficient



Response Comparision

- Over damped response is BIG and slow
- Critically damped response is small and FAST



Response: *Case 3 Under Damped*

- *Case 3:* $\sigma^2 - \rho < 0,\ b < 2\sqrt{km}$ ⇨ Complex Conjugate pair of poles

- *System poles* $\alpha_1, \alpha_2 = -\sigma \pm j\omega$

- *Response*

(3.52),

$$\begin{aligned} Y(s) &= \frac{y(0)s + [2\sigma y(0) + y'(0)]}{(s+\sigma-j\omega)(s+\sigma+j\omega)} + \frac{\eta}{(s+\sigma-j\omega)(s+\sigma+j\omega)}F(s) \\ &= \frac{K_1 s + K_2}{(s+\sigma)^2 - (j\omega)^2} + \frac{\eta}{(s+\sigma)^2 - (j\omega)^2}F(s) \\ &= K_1\frac{s}{(s+\sigma)^2+\omega^2} + K_2\frac{1}{(s+\sigma)^2+\omega^2} + \eta\frac{1}{(s+\sigma)^2+\omega^2}\frac{A}{s} \\ &= \overbrace{K_1\left[\frac{s+\sigma}{(s+\sigma)^2+\omega^2} - \frac{\sigma}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}\right]} + \frac{K_2}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2} \\ &\quad \frac{\eta}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}\frac{A}{s} \qquad (3.65) \end{aligned}$$

---

but *s* => *(s+σ)* → exponential scaling → $e^{-\sigma t}\cos \omega t$

↑

$\cos \omega t$

↑

$$= K_1\left[\frac{s+\sigma}{(s+\sigma)^2+\omega^2} - \frac{\sigma}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}\right] + \frac{K_2}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}$$

$$\frac{\eta}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}\frac{A}{s} \qquad (3.65)$$

$\sin \omega t$

↓

but *s* => *(s+σ)* → exponential scaling

↓

$e^{-\sigma t}\sin \omega t$

↓

$e^{-\sigma t}\sin \omega t$

↓

$\int_0^t e^{-\sigma t}\sin \omega t dt$

Response : Case 3 Under Damped

$$\begin{aligned} y(t) &= K_1 e^{-\sigma t}\cos\omega t - \frac{K_1\sigma}{\omega}e^{-\sigma t}\sin\omega t + \frac{K_2}{\omega}e^{-\sigma t}\sin\omega t \\ &\quad + \frac{\eta A}{\omega}\int_0^t e^{-\sigma t}\sin\omega t dt \\ &= e^{-\sigma t}\left\{K_1\cos\omega t + \frac{(K_2 - \sigma K_1)}{\omega}\sin\omega t\right\} \\ &\quad + \frac{\eta A}{\omega}\left\{\frac{\omega}{\omega^2+\sigma^2} - e^{-\sigma t}\sin(\omega t + \phi_E)\right\} \end{aligned}$$

- Decaying sinusoidal indicates an oscillation, which is a result of weaker damper to resist the speed adequately

k=125N/cm b=700Ns/cm 1

↓

k =2450[N/cm] ← 2

↓

k =2450[N/cm] b = 300 Ns/cm 3

$f/k = 500/2450 = 0.2cm$

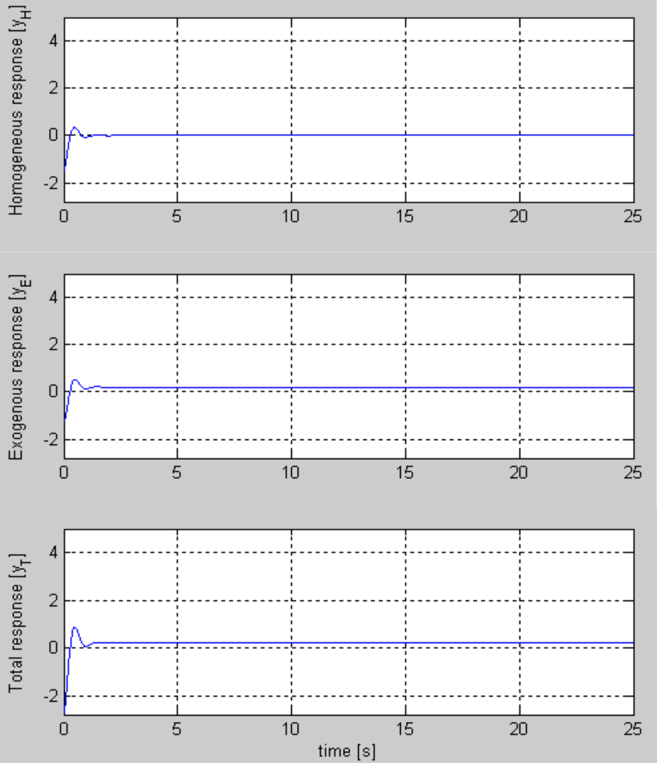
Steady state response

Under Damped Response

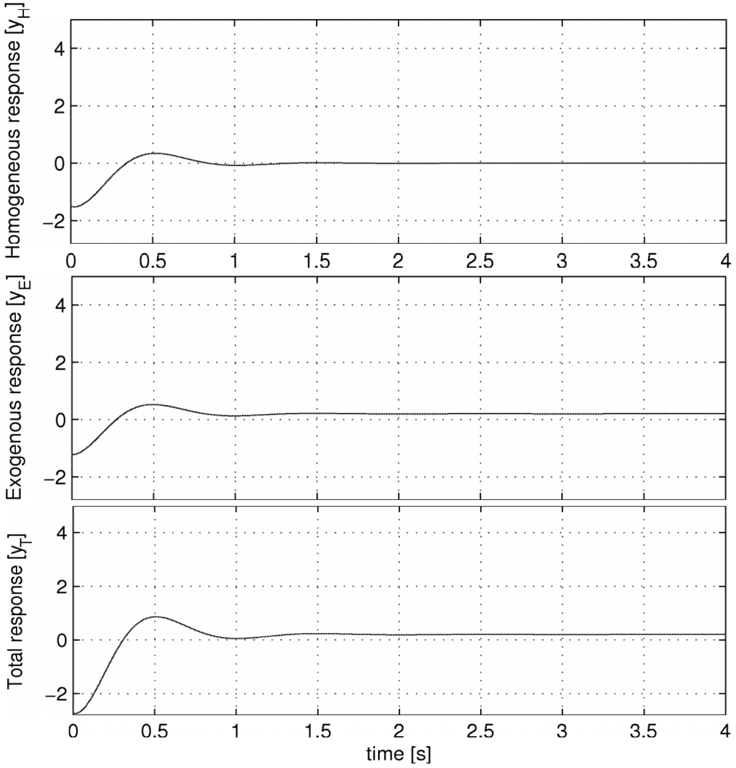
- Poles

  $\alpha_1, \alpha_2 = -3 \pm j6.3$

- Oscillatory due to spring action being dominant
- Oscillations die out, response is stable

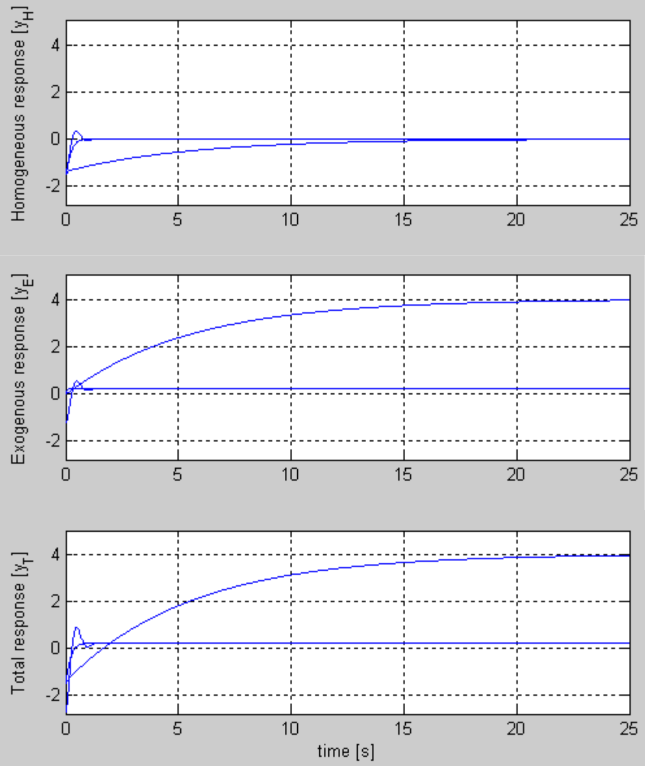


Oscillatory Response



Response Comparison

- Critical damping and under damping responses are better than over damping response
- Overshoot can be a problem in motion control systems (robots), however, it is acceptable in process control systems (temperature, pressure)
- Under damping response is the fastest of all



Poles and Response

Over damped

Critically damped

Under damped